Randomized Sensor Selection
in Sequential Hypothesis Testing

Vaibhav Srivastava   Kurt Plarre   Francesco Bullo

Abstract—We consider the problem of sensor selection for time-optimal detection of a hypothesis. We consider a group of sensors transmitting their observations to a fusion center. The fusion center considers the output of only one randomly chosen sensor at the time, and performs a sequential hypothesis test. We consider the class of sequential tests which are easy to implement, asymptotically optimal, and computationally amenable. For three distinct performance metrics, we show that, for a generic set of sensors and binary hypothesis, the fusion center needs to consider at most two sensors. We also show that for the case of multiple hypothesis, the optimal policy needs at most as many sensors to be observed as the number of underlying hypotheses.

Index Terms—Sensor selection, decision making, SPRT, MSPRT, sequential hypothesis testing, linear-fractional programming.

I. INTRODUCTION

In today's information-rich world, different sources are best informers about different topics. If the topic under consideration is well known beforehand, then one chooses the best source. Otherwise, it is not obvious what source or how many sources one should observe. This need to identify sensors (information sources) to be observed in decision making problems is found in many common situations, e.g., when deciding which news channel to follow. When a person decides what information source to follow, she relies in general upon her experience, i.e., one knows through experience what combination of news channels to follow.

In engineering applications, a reliable decision on the underlying hypothesis is made through repeated measurements. Given infinitely many observations, decision making can be performed accurately. Given a cost associated to each observation, a well-known tradeoff arises between accuracy and number of iterations. Various sequential hypothesis tests have been proposed to detect the underlying hypothesis within a given degree of accuracy. There exist two different classes of sequential tests. The first class includes sequential tests developed from the dynamic programming point of view. These tests are optimal and, in general, difficult to implement [5]. The second class consists of easily-implementable and asymptotically-optimal sequential tests; a widely-studied example is the Sequential Probability Ratio Test (SPRT) for binary hypothesis testing and its extension, the Multi-hypothesis Sequential Probability Ratio Test (MSPRT).

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In this paper, we consider the problem of quickest decision making using sequential probability ratio tests. Recent advances in cognitive psychology [7] show that the performance of a human performing decision making tasks, such as "two-alternative forced choice tasks," is well modeled by a drift diffusion process, i.e., the continuous-time version of SPRT. Roughly speaking, modeling decision making as an SPRT process is somehow appropriate even for situations in which a human is making the decision.

Sequential hypothesis testing and quickest detection problems have been vastly studied [17], [4]. The SPRT for binary decision making was introduced by Wald in [21], and was extended by Armitage to multiple hypothesis testing in [1]. The Armitage test, unlike the SPRT, is not necessarily optimal [24]. Various other tests for multiple hypothesis testing have been developed throughout the years; a survey is presented in [18]. Designing hypothesis tests, i.e., choosing thresholds to decide within a given expected number of iterations, through any of the procedures in [18] is infeasible as none of them provides any results on the expected sample size. A sequential test for multiple hypothesis testing was developed in [5], [10], and [11], which provides with an asymptotic expression for the expected sample size. This sequential test is called the MSPRT and reduces to the SPRT in case of binary hypothesis.

Recent years have witnessed a significant interest in the problem of sensor selection for optimal detection and estimation. Tay et al [20] discuss the problem of censoring sensors for decentralized binary detection. They assess the quality of sensor data by the Neyman-Pearson and a Bayesian binary hypothesis test and decide on which sensors should transmit their observation at that time instant. Gupta et al [13] focus on stochastic sensor selection and minimize the error covariance of a process estimation problem. Isler et al [14] propose geometric sensor selection schemes for error minimization in target detection. Debouk et al [9] formulate a Markovian decision problem to ascertain some property in a dynamical system, and choose sensors to minimize the associated cost. Wang et al [22] design entropy-based sensor selection algorithms for target localization. Joshi et al [15] present a convex optimization-based heuristic to select multiple sensors for optimal parameter estimation. Bajović et al [3] discuss sensor selection problems for Neyman-Pearson binary hypothesis testing in wireless sensor networks.

A third and last set of references related to this paper are those on linear-fractional programming. Various iterative and cumbersome algorithms have been proposed to optimize linear-fractional functions [8], [2]. In particular, for the problem of minimizing the sum and the maximum of linear-

fractional functionals, some efficient iterative algorithms have been proposed, including the algorithms by Falk et al [12] and by Benson [6].

In this paper, we analyze the problem of time-optimal sequential decision making in the presence of multiple switching sensors and determine a sensor selection strategy to achieve the same. We consider a sensor network where all sensors are connected to a fusion center. The fusion center, at each instant, receives information from only one sensor. Such a situation arises when we have interfering sensors (e.g., sonar sensors), a fusion center with limited attention or information processing capabilities, or sensors with shared communication resources. The fusion center implements a sequential hypothesis test with the gathered information. We consider two such tests, namely, the SPRT and the MSPRT for binary and multiple hypothesis, respectively. First, we develop a version of the SPRT and the MSPRT where the sensor is randomly switched at each iteration, and determine the expected time that these tests require to obtain a decision within a given degree of accuracy. Second, we identify the set of sensors that minimize the expected decision time. We consider three different cost functions, namely, the conditioned decision time, the worst case decision time, and the average decision time. We show that the expected decision time, conditioned on a given hypothesis, using these sequential tests is a linear-fractional function defined on the probability simplex. We exploit the special structure of our domain (probability simplex), and the fact that our data is positive to tackle the problem of the sum and the maximum of linear-fractional functionals analytically. Our approach provides insights into the behavior of these functions. The major contributions of this paper are:

i) We develop a version of the SPRT and the MSPRT where the sensor is selected randomly at each observation.

ii) We determine the asymptotic expressions for the thresholds and the expected sample size for these sequential tests.

iii) We incorporate the processing time of the sensors into these models to determine the expected decision time.

iv) We show that, to minimize the conditioned expected decision time, the optimal policy requires only one sensor to be observed.

v) We show that, for a generic set of sensors and M underlying hypotheses, the optimal average decision time policy requires the fusion center to consider at most M sensors.

vi) For the binary hypothesis case, we identify the optimal set of sensors in the worst case and the average decision time minimization problems. Moreover, we determine an optimal probability distribution for the sensor selection.

vii) In the worst case and the average decision time minimization problems, we encounter the problem of minimization of sum and maximum of linear-fractional functionals. We treat these problems analytically, and provide insight into their optimal solutions.

The remainder of the paper is organized in following way. In Section II, we present the problem setup. Some preliminaries are presented in Section III. We develop the switching-sensor version of the SPRT and the MSPRT procedures in Section IV. In Section V, we formulate the optimization problems for time-optimal sensor selection, and determine their solution. We elucidate the results obtained through numerical examples in Section VI. Our concluding remarks are in Section VII.

II. PROBLEM SETUP

We consider a group of $n > 1$ agents (e.g., robots, sensors, or cameras), which take measurements and transmit them to a fusion center. We generically call these agents "sensors." We identify the fusion center with a person supervising the agents, and call it "the supervisor."
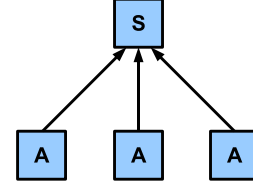

Fig. 1. The agents A transmit their observation to the supervisor S, one at the time. The supervisor performs a sequential hypothesis test to decide on the underlying hypothesis.

The goal of the supervisor is to decide, based on the measurements it receives, which of the $M \geq 2$ alternative hypotheses or "states of nature," H_k, $k \in \{0, \ldots, M-1\}$ is correct. For doing so, the supervisor uses sequential hypothesis tests, which we briefly review in the next section.

We assume that only one sensor can transmit to the supervisor at each (discrete) time instant. Equivalently, the supervisor can process data from only one of the n agents at each time. Thus, at each time, the supervisor must decide which sensor should transmit its measurement. We are interested in finding the optimal sensor(s), which must be observed in order to minimize the decision time.

We model the setup in the following way:

i) Let $s_l \in \{1, \ldots, n\}$ indicate which sensor transmits its measurement at time instant $l \in \mathbb{N}$.

ii) Conditioned on the hypothesis H_k, $k \in \{0, \ldots, M-1\}$, the probability that the measurement at sensor s is y, is denoted by $f_s^k(y)$.

iii) The prior probability of the hypothesis H_k, $k \in \{0, \ldots, M-1\}$, being correct is π_k.

iv) The measurement of sensor s at time l is y_{s_l}. We assume that, conditioned on hypothesis H_k, y_{s_l} is independent of $y(s_{\bar{l}})$, for $(l, s_l) \neq (\bar{l}, s_{\bar{l}})$.

v) The time it takes for sensor s to transmit its measurement (or for the supervisor to process it) is $T_s > 0$.

vi) The supervisor chooses a sensor randomly at each time instant; the probability to choose sensor s is stationary and given by q_s.

vii) The supervisor uses the data collected to execute a sequential hypothesis test with the desired probability of incorrect decision, conditioned on hypothesis H_k, given by α_k.

viii) We assume that there are no two sensors with identical conditioned probability distribution $f_s^k(y)$ and processing time T_s. If there are such sensors, we club them together in a single node, and distribute the probability assigned to that node equally among them.

III. PRELIMINARIES

A. Linear-fractional function

Given parameters $A \in \mathbb{R}^{q \times p}$, $B \in \mathbb{R}^q$, $c \in \mathbb{R}^p$, and $d \in \mathbb{R}$, the function $g : \{z \in \mathbb{R}^p \mid c^T z + d > 0\} \rightarrow \mathbb{R}^q$, defined by

$$g(x) = \frac{Ax + B}{c^T x + d},$$

is called a *linear-fractional function* [8]. A linear-fractional function is quasi-convex as well as quasi-concave. In particular, if $q = 1$, then any scalar linear-fractional function g satisfies

$$\begin{aligned} g(tx + (1-t)y) &\leq \max\{g(x), g(y)\}, \\ g(tx + (1-t)y) &\geq \min\{g(x), g(y)\}, \end{aligned} \tag{1}$$

for all $t \in [0, 1]$ and $x, y \in \{z \in \mathbb{R}^p \mid c^T z + d > 0\}$.

B. Kullback-Leibler distance

Given two probability distributions functions $f_1 : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ and $f_2 : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$, the Kullback-Leibler distance $\mathcal{D} : \mathcal{L}^1 \times \mathcal{L}^1 \rightarrow \mathbb{R}$ is defined by

$$\mathcal{D}(f_1, f_2) = \mathbb{E}\left[\log \frac{f_1(X)}{f_2(X)}\right] = \int_{\mathbb{R}} f_1(x) \log \frac{f_1(x)}{f_2(x)} dx.$$

Further, $\mathcal{D}(f_1, f_2) \geq 0$, and the equality holds if and only if $f_1 = f_2$ almost everywhere.

C. Sequential Probability Ratio Test

The SPRT is a sequential binary hypothesis test that provides us with two thresholds to decide on some hypothesis, opposed to classical hypothesis tests, where we have a single threshold. Consider two hypothesis H_0 and H_1, with prior probabilities π_0 and π_1, respectively. Given their conditional probability distribution functions $f(y|H_0) = f^0(y)$ and $f(y|H_1) = f^1(y)$, and repeated measurements $\{y_1, y_2, \ldots\}$, with λ_0 defined by

$$\lambda_0 = \log \frac{\pi_1}{\pi_0}, \tag{2}$$

the SPRT provides us with two constants η_0 and η_1 to decide on a hypothesis at each time instant l, in the following way:

i) Compute the log likelihood ratio: $\lambda_l := \log \frac{f^1(y_l)}{f^0(y_l)}$,

ii) Integrate evidence up to time N, i.e., $\Lambda_N := \sum_{l=0}^{N} \lambda_l$,

iii) Decide only if a threshold is crossed, i.e.,

$$\begin{cases} \Lambda_N > \eta_1, & \text{say } H_1, \\ \Lambda_N < \eta_0, & \text{say } H_0, \\ \Lambda_N \in \,]\eta_0, \eta_1[, & \text{continue sampling.} \end{cases}$$

Given the probability of false alarm $\mathbb{P}(H_1|H_0) = \alpha_0$ and probability of missed detection $\mathbb{P}(H_0|H_1) = \alpha_1$, the Wald's thresholds η_0 and η_1 are defined by

$$\eta_0 = \log \frac{\alpha_1}{1 - \alpha_0}, \quad \text{and} \quad \eta_1 = \log \frac{1 - \alpha_1}{\alpha_0}. \tag{3}$$

The expected sample size N, for decision using SPRT is asymptotically given by

$$\begin{aligned} \mathbb{E}[N|H_0] &\rightarrow -\frac{(1-\alpha_0)\eta_0 + \alpha_0 \eta_1 - \lambda_0}{\mathcal{D}(f^0, f^1)}, \quad \text{and} \\ \mathbb{E}[N|H_1] &\rightarrow \frac{\alpha_1 \eta_0 + (1-\alpha_1)\eta_1 - \lambda_0}{\mathcal{D}(f^1, f^0)}, \end{aligned} \tag{4}$$

as $-\eta_0, \eta_1 \rightarrow \infty$. The asymptotic expected sample size expressions in equation (4) are valid for large thresholds. The use of these asymptotic expressions as approximate expected sample size is a standard approximation in the information theory literature, and is known as Wald's approximation [4], [17], [19].

For given error probabilities, the SPRT is the optimal sequential binary hypothesis test, if the sample size is considered as the cost function [19].

D. Multi-hypothesis Sequential Probability Ratio Test

The MSPRT for multiple hypothesis testing was introduced in [5], and was further generalized in [10] and [11]. It is described as follows. Given M hypotheses with their prior probabilities $\pi_k, k \in \{0, \ldots, M-1\}$, the posterior probability after N observations $y_l, l \in \{1, \ldots, N\}$ is given by

$$p_N^k = \mathbb{P}(H = H_k | y_1, \ldots, y_N) = \frac{\pi_k \prod_{l=1}^{N} f^k(y_l)}{\sum_{j=1}^{M-1} \pi_j \left(\prod_{l=1}^{n} f^j(y_l)\right)},$$

where f^k is the probability density function of the observation of the sensor, conditioned on hypothesis k.

Before we state the MSPRT, for a given N, we define $\bar{k}$ by

$$\bar{k} = \operatorname*{argmax}_{j \in \{0, \ldots, M-1\}} \pi_j \prod_{l=1}^{N} f^j(y_l).$$

The MSPRT at each sampling iteration l is defined as

$$\begin{cases} p_l^k > \frac{1}{1+\eta_k}, \text{for at least one } k, & \text{say } H_{\bar{k}}, \\ \text{otherwise}, & \text{continue sampling}, \end{cases}$$

where the thresholds η_k, for given frequentist error probabilities (accept a given hypothesis wrongly) $\alpha_k, k \in \{0, \ldots, M-1\}$, are given by

$$\eta_k = \frac{\alpha_k}{\pi_k \gamma_k}, \tag{5}$$

where $\gamma_k \in \,]0, 1[$ is a constant function of f^k (see [5]).

It can be shown [5] that the expected sample size of the MSPRT, conditioned on a hypothesis, satisfies

$$\mathbb{E}[N|H_k] \rightarrow \frac{-\log \eta_k}{\delta_k}, \quad \text{as} \quad \max_{k \in \{0, \ldots, M-1\}} \eta_k \rightarrow 0^+,$$

where $\delta_k = \min\{\mathcal{D}(f^k, f^j) \mid j \in \{0, \ldots, M-1\} \setminus \{k\}\}$.

The MSPRT is an easily-implementable hypothesis test and is shown to be asymptotically optimal in [5], [10].

IV. SEQUENTIAL HYPOTHESIS TESTS WITH SWITCHING SENSORS

A. SPRT with switching sensors

Consider the case when the fusion center collects data from n sensors. At each iteration the fusion center looks at one sensor chosen randomly with probability q_s, $s \in \{1, \ldots, n\}$. The fusion center performs SPRT with the collected data. We define this procedure as SPRT with switching sensors. If we assume that sensor s_l is observed at iteration l, and the observed value is y_{s_l}, then SPRT with switching sensors is described as following, with the thresholds η_0 and η_1 defined in equation (3), and λ_0 defined in equation (2):

i) Compute log likelihood ratio:

$$\lambda_l := \log \frac{f^1_{s_l}(y_{s_l})}{f^0_{s_l}(y_{s_l})},$$

ii) Integrate evidence up to time N, i.e., $\Lambda_N := \sum_{l=0}^{N} \lambda_l$,

iii) Decide only if a threshold is crossed, i.e.,

$$\begin{cases} \Lambda_N > \eta_1, & \text{say } H_1, \\ \Lambda_N < \eta_0, & \text{say } H_0, \\ \Lambda_N \in \,]\eta_0, \eta_1[, & \text{continue sampling.} \end{cases}$$

Lemma 1 (Expected sample size): For the SPRT with switching sensors described above, the expected sample size conditioned on a hypothesis is asymptotically given by:

$$\begin{aligned} \mathbb{E}\,[N|H_0] &\to -\frac{(1-\alpha_0)\eta_0 + \alpha_0\eta_1 - \lambda_0}{\sum_{s=1}^{n} q_s \mathcal{D}(f^0_s, f^1_s)}, \quad \text{and} \\ \mathbb{E}\,[N|H_1] &\to \frac{\alpha_1\eta_0 + (1-\alpha_1)\eta_1 - \lambda_0}{\sum_{s=1}^{n} q_s \mathcal{D}(f^1_s, f^0_s)}, \end{aligned} \tag{6}$$

as $-\eta_0, \eta_1 \to \infty$.

Proof: Similar to the proof of Theorem 3.2 in [23]. ∎

The expected sample size converges to the values in equation (6) for large thresholds. From equation (3), it follows that large thresholds correspond to small error probabilities. In the remainder of the paper, we assume that the error probabilities are chosen small enough, so that the above asymptotic expression for sample size is close to the actual expected sample size.

Lemma 2 (Expected decision time): Given the processing time of the sensors $T_s, s \in \{1, \ldots, n\}$, the expected decision time of the SPRT with switching sensors T_d, conditioned on the hypothesis H_k, $k \in \{0, 1\}$, is

$$\mathbb{E}\,[T_d|H_k] = \frac{\sum_{s=1}^{n} q_s T_s}{\sum_{s=1}^{n} q_s I^k_s} = \frac{q \cdot T}{q \cdot I^k}, \quad \text{for each } k \in \{0, 1\}, \tag{7}$$

where $T, I^k \in \mathbb{R}^n_{>0}$, are constant vectors for each $k \in \{0, 1\}$.

Proof: The decision time using SPRT with switching sensors is the sum of sensor's processing time at each iteration. We observe that the number of iterations in SPRT and

the processing time of sensors are independent. Hence, the expected value of the decision time T_d is

$$\mathbb{E}\,[T_d|H_k] = \mathbb{E}\,[N|H_k]\,\mathbb{E}\,[T], \quad \text{for each } k \in \{0, 1\}. \tag{8}$$

By the definition of expected value,

$$\mathbb{E}\,[T] = \sum_{s=1}^{n} q_s T_s. \tag{9}$$

From equations (6), (8), and (9) it follows that

$$\mathbb{E}\,[T_d|H_k] = \frac{\sum_{s=1}^{n} q_s T_s}{\sum_{s=1}^{n} q_s I^k_s} = \frac{q \cdot T}{q \cdot I^k}, \quad \text{for each } k \in \{0, 1\},$$

where $I^k_s \in \mathbb{R}_{>0}$ is a constant, for each $k \in \{0, 1\}$, and $s \in \{1, \ldots, n\}$. ∎

B. MSPRT with switching sensors

We call the MSPRT with the data collected from n sensors while observing only one sensor at a time as the MSPRT with switching sensors. The one sensor to be observed at each time is determined through a randomized policy, and the probability of choosing sensor s is stationary and given by q_s. Assume that the sensor $s_l \in \{1, \ldots, n\}$ is chosen at time instant l, and the prior probabilities of the hypothesis are given by $\pi_k, k \in \{0, \ldots, M-1\}$, then the posterior probability after N observations $y_l, l \in \{1, \ldots, N\}$ is given by

$$p^k_N = \mathbb{P}(H_k|y_1, \ldots, y_N) = \frac{\pi_k \prod_{l=1}^{N} f^k_{s_l}(y_l)}{\sum_{j=0}^{M-1} \pi_j \left(\prod_{l=1}^{N} f^j_{s_l}(y_l) \right)},$$

Before we state the MSPRT with switching sensors, for a given N, we define $\tilde{k}$ by

$$\tilde{k} = \underset{k \in \{0, \ldots, M-1\}}{\operatorname{argmax}} \ \pi_k \prod_{l=1}^{N} f^k_{s_l}(y_l).$$

For the thresholds η_k, $k \in \{0, \ldots, M-1\}$, defined in equation (5), the MSPRT with switching sensors at each sampling iteration N is defined by

$$\begin{cases} p^k_n > \frac{1}{1+\eta_k}, \text{ for at least one } k, & \text{say } H_{\tilde{k}}, \\ \text{otherwise}, & \text{continue sampling.} \end{cases}$$

Before we state the results on asymptotic sample size and expected decision time, we introduce the following notation. For a given hypothesis H_k, and a sensor s, we define $j^*_{(s,k)}$ by

$$j^*_{(s,k)} = \underset{\substack{j \in \{0, \ldots, M-1\} \\ j \neq k}}{\operatorname{argmin}} \ \mathcal{D}(f^k_s, f^j_s).$$

We also define $\mathcal{E}_\mathcal{D} : \Delta_{n-1} \times (\mathcal{L}^1)^n \times (\mathcal{L}^1)^n \to \mathbb{R}$ by

$$\mathcal{E}_\mathcal{D}(q, f^k, f^{j^*_k}) = \sum_{s=1}^{n} q_s \mathcal{D}(f^k_s, f^{j^*_{(s,k)}}_s),$$

where Δ_{n-1} represents the probability simplex in $\mathbb{R}^n$.

Lemma 3 (Expected sample size): Given thresholds η_j, $j \in \{0, \ldots, M-1\}$, the sample size N required for decision satisfies

$$\frac{\mathbb{E}[N|H_k]}{-\log \eta_k} \to \frac{1}{\mathcal{E}_{\mathcal{D}}(q, f^k, f^{j_k^*})},$$

as $\max_{j \in \{0,\ldots,M-1\}} \eta_j \to 0$.

Proof: The proof follows from Theorem 5.1 of [5] and the observation that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{l=1}^{N} \log \frac{f_{s_l}^k(X_l)}{f_{s_l}^j(X_l)} = \sum_{s=1}^{n} q_s \mathcal{D}(f_s^k, f_s^j).$$

∎

Lemma 4 (Expected decision time): Given the processing time of the sensors $T_s, s \in \{1, \ldots, n\}$, the expected decision time T_d conditioned on the hypothesis H_k, for each $k \in \{0, \ldots, M-1\}$, is given by

$$\mathbb{E}[T_d|H_k] = \frac{-\log \eta_k}{\mathcal{E}_{\mathcal{D}}(q, f^k, f^{j_k^*})} \sum_{s=1}^{n} q_s T_s = \frac{q \cdot T}{q \cdot \bar{I}^k}, \quad (10)$$

where $T, \bar{I}^k \in \mathbb{R}_{>0}^n$ are constants.

Proof: Similar to the proof of Lemma 2. ∎

V. OPTIMAL SENSOR SELECTION

In this section we consider sensor selection problems with the aim to minimize the expected decision time of a sequential hypothesis test with switching sensors. As exemplified in Lemma 4, the problem features multiple conditioned decision times and, therefore, multiple distinct cost functions are of interest. In Scenario I below, we aim to minimize the decision time conditioned upon one specific hypothesis being true; in Scenarios II and III we will consider worst-case and average decision times. In all three scenarios the decision variables take values in the probability simplex.

Minimizing decision time conditioned upon a specific hypothesis may be of interest when fast reaction is required in response to the specific hypothesis being indeed true. For example, in change detection problems one aims to quickly detect a change in a stochastic process; the CUSUM algorithm (also referred to as Page's test) [16] is widely used in such problems. It is known [4] that, with fixed threshold, the CUSUM algorithm for quickest change detection is equivalent to an SPRT on the observations taken after the change has occurred. We consider the minimization problem for a single conditioned decision time in Scenario I below and we show that, in this case, observing the best sensor each time is the optimal strategy.

In general, no specific hypothesis might play a special role in the problem and, therefore, it is of interest to simultaneously minimize multiple decision times over the probability simplex. This is a multi-objective optimization problem, and may have Pareto-optimal solutions. We tackle this problem by constructing a single aggregate objective function. In the binary hypothesis case, we construct two single aggregate objective functions as the maximum and the average of the two conditioned decision times. These two functions are discussed in Scenario II and Scenario III respectively. In the multiple

hypothesis setting, we consider the single aggregate objective function constructed as the average of the conditioned decision times. An analytical treatment of this function for $M > 2$, is difficult. We determine the optimal number of sensors to be observed, and direct the interested reader to some iterative algorithms to solve such optimization problems. This case is also considered under Scenario III.

Before we pose the problem of optimal sensor selection, we introduce the following notation. We denote the probability simplex in $\mathbb{R}^n$ by Δ_{n-1}, and the vertices of the probability simplex Δ_{n-1} by e_i, $i \in \{1, \ldots, n\}$. We refer to the line joining any two vertices of the simplex as an *edge*. Finally, we define $g^k : \Delta_{n-1} \to \mathbb{R}$, $k \in \{0, \ldots, M-1\}$, by

$$g^k(q) = \frac{q \cdot T}{q \cdot I^k}.$$

A. Scenario I (Optimization of conditioned decision time):

We consider the case when the supervisor is trying to detect a particular hypothesis, irrespective of the present hypothesis. The corresponding optimization problem for a fixed $k \in \{0, \ldots, M-1\}$ is posed in the following way:

$$\begin{aligned} \text{minimize} \quad & g^k(q) \\ \text{subject to} \quad & q \in \Delta_{n-1}. \end{aligned} \quad (11)$$

The solution to this minimization problem is given in the following theorem.

Theorem 1 (Optimization of conditioned decision time): The solution to the minimization problem (11) is $q^* = e_{s^*}$, where s^* is given by

$$s^* = \operatorname*{argmin}_{s \in \{1,\ldots,n\}} \frac{T_s}{I_s^k},$$

and the minimum objective function is

$$\mathbb{E}[T_d^*|H_k] = \frac{T_{s^*}}{I_{s^*}^k}. \quad (12)$$

Proof: We notice that objective function is a linear-fractional function. In the following argument, we show that the minima occurs at one of the vertices of the simplex.

We first notice that the probability simplex is the convex hull of the vertices, i.e., any point $\tilde{q}$ in the probability simplex can be written as

$$\tilde{q} = \sum_{s=1}^{n} \alpha_s e_s, \qquad \sum_{s=1}^{n} \alpha_s = 1, \quad \text{and} \quad \alpha_s \geq 0.$$

We invoke equation (1), and observe that for some $\beta \in [0, 1]$ and for any $s, r \in \{1, \ldots, n\}$

$$g^k(\beta e_s + (1-\beta)e_r) \geq \min\{g^k(e_s), g^k(e_r)\}, \quad (13)$$

which can be easily generalized to

$$g^k(\tilde{q}) \geq \min_{l \in \{1,\ldots,n\}} g^k(e_s), \quad (14)$$

for any point $\tilde{q}$ in the probability simplex Δ_{n-1}. Hence, minima will occur at one of the vertices e_{s^*}, where s^* is given by

$$s^* = \operatorname*{argmin}_{s \in \{1,\ldots,n\}} g^k(e_s) = \operatorname*{argmin}_{s \in \{1,\ldots,n\}} \frac{T_s}{I_s^k}.$$

∎

B. Scenario II (Optimization of the worst case decision time):

For the binary hypothesis testing, we consider the multi-objective optimization problem of minimizing both decision times simultaneously. We construct single aggregate objective function by considering the maximum of the two objective functions. This turns out to be a worst case analysis, and the optimization problem for this case is posed in the following way:

$$
\begin{aligned}
\text{minimize} \quad & \max\left\{g^0(q), g^1(q)\right\}, \\
\text{subject to} \quad & q \in \Delta_{n-1}.
\end{aligned}
\tag{15}
$$

Before we move on to the solution of above minimization problem, we state the following results.

Lemma 5 (Monotonicity of conditioned decision times): The functions g^k, $k \in \{0, \ldots, M-1\}$ are monotone on the probability simplex Δ_{n-1}, in the sense that given two points $q_1, q_2 \in \Delta_{n-1}$, the function g^k is monotonically non-increasing or monotonically non-decreasing along the line joining q_1 and q_2.

Proof: Consider probability vectors $q_1, q_2 \in \Delta_{n-1}$. Any point q on line joining q_1 and q_2 can be written as $q(t) = tq_1 + (1-t)q_2$, $t \in \,]0,1[$. We note that $g^k(q(t))$ is given by:

$$
g^k(q(t)) = \frac{t(q_1 \cdot T) + (1-t)(q_2 \cdot T)}{t(q_1 \cdot I^k) + (1-t)(q_2 \cdot I^k)}.
$$

The derivative of g^k along the line joining q_1 and q_2 is given by

$$
\frac{d}{dt}g^k(q(t)) = \left(g^k(q_1) - g^k(q_2)\right)
$$
$$
\times \frac{(q_1 \cdot I^k)(q_2 \cdot I^k)}{(t(q_1 \cdot I^k) + (1-t)(q_2 \cdot I^k))^2}.
$$

We note that the sign of the derivative of g^k along the line joining two points q_1, q_2 is fixed by the choice of q_1 and q_2. Hence, the function g^k is monotone over the line joining q_1 and q_2. Moreover, note that if $g^k(q_1) \neq g^k(q_2)$, then g^k is strictly monotone. Otherwise, g^k is constant over the line joining q_1 and q_2. ∎

Lemma 6 (Location of min-max): Define $g : \Delta_{n-1} \to \mathbb{R}_{\geq 0}$ by $g = \max\{g^0, g^1\}$. A minimum of g lies at the intersection of the graphs of g^0 and g^1, or at some vertex of the probability simplex Δ_{n-1}.

Proof: Case 1: The graphs of g^0 and g^1 do not intersect at any point in the simplex Δ_{n-1}.

In this case, one of the functions g^0 and g^1 is an upper bound to the other function at every point in the probability simplex Δ_{n-1}. Hence, $g = g^k$, for some $k \in \{0, 1\}$, at every point in the probability simplex Δ_{n-1}. From Theorem 1, we know that the minima of g^k on the probability simplex Δ_{n-1} lie at some vertex of the probability simplex Δ_{n-1}.

Case 2: The graphs of g^0 and g^1 intersect at a set Q in the probability simplex Δ_{n-1}, and let $\bar{q}$ be some point in the set Q.

Suppose, a minimum of g occurs at some point $q^* \in \text{relint}(\Delta_{n-1})$, and $q^* \notin Q$, where $\text{relint}(\cdot)$ denotes the relative interior. With out loss of generality, we can assume that $g^0(q^*) > g^1(q^*)$. Also, $g^0(\bar{q}) = g^1(\bar{q})$, and $g^0(q^*) < g^0(\bar{q})$ by assumption.

We invoke Lemma 5, and notice that g^0 and g^1 can intersect at most once on a line. Moreover, we note that $g^0(q^*) > g^1(q^*)$, hence, along the half-line from $\bar{q}$ through q^*, $g^0 > g^1$, that is, $g = g^0$. Since $g^0(q^*) < g^0(\bar{q})$, g is decreasing along this half-line. Hence, g should achieve its minimum at the boundary of the simplex Δ_{n-1}, which contradicts that q^* is in the relative interior of the simplex Δ_{n-1}. In summary, if a minimum of g lies in the relative interior of the probability simplex Δ_{n-1}, then it lies at the intersection of the graphs of g^0 and g^1.

The same argument can be applied recursively to show that if a minimum lies at some point $q^\dagger$ on the boundary, then either $g^0(q^\dagger) = g^1(q^\dagger)$ or the minimum lies at the vertex. ∎

In the following arguments, let Q be the set of points in the simplex Δ_{n-1}, where $g^0 = g^1$, that is,

$$
Q = \{q \in \Delta_{n-1} \mid q \cdot (I^0 - I^1) = 0\}.
\tag{16}
$$

Also notice that the set Q is non empty if and only if $I^0 - I^1$ has at least one non-negative and one non-positive entry. If the set Q is empty, then it follows from Lemma 6 that the solution of optimization problem in equation (15) lies at some vertex of the probability simplex Δ_{n-1}. Now we consider the case when Q is non empty. We assume that the sensors have been re-ordered such that the entries in $I^0 - I^1$ are in ascending order. We further assume that, for $I^0 - I^1$, the first m entries, $m < n$, are non positive, and the remaining entries are positive.

Lemma 7 (Intersection polytope): If the set Q defined in equation (16) is non empty, then the polytope generated by the points in the set Q has vertices given by:

$$
\tilde{Q} = \{\tilde{q}^{sr} \mid s \in \{1, \ldots, m\} \text{ and } r \in \{m+1, \ldots, n\}\},
$$

where for each $i \in \{1, \ldots, n\}$

$$
\tilde{q}_i^{sr} = \begin{cases} \dfrac{(I_r^0 - I_r^1)}{(I_r^0 - I_r^1) - (I_s^0 - I_s^1)}, & \text{if} \quad i = s, \\ 1 - \tilde{q}_s^{sr}, & \text{if} \quad i = r, \\ 0, & \text{otherwise.} \end{cases}
\tag{17}
$$

Proof: Any $q \in Q$ satisfies the following constraints

$$
\sum_{s=1}^n q_s = 1, \quad q_s \in [0, 1],
\tag{18}
$$

$$
\sum_{s=1}^n q_s(I_s^0 - I_s^1) = 0,
\tag{19}
$$

Eliminating q_n, using equation (18) and equation (19), we get:

$$
\sum_{s=1}^{n-1} \beta_s q_s = 1, \quad \text{where} \quad \beta_s = \frac{(I_n^0 - I_n^1) - (I_s^0 - I_s^1)}{(I_n^0 - I_n^1)}.
\tag{20}
$$

The equation (20) defines a hyperplane, whose extreme points in $\mathbb{R}_{\geq 0}^{n-1}$ are given by

$$
\tilde{q}^{sn} = \frac{1}{\beta_s} e_s, \quad i \in \{1, \ldots, n-1\}.
$$

Note that for $s \in \{1, \ldots, m\}$, $\tilde{q}^{sn} \in \Delta_{n-1}$. Hence, these points define some vertices of the polytope generated by points in the set Q. Also note that the other vertices of the polytope

can be determined by the intersection of each pair of lines through $\tilde{q}^{sn}$ and $\tilde{q}^{rn}$, and e_s and e_r, for $s \in \{1, \ldots, m\}$, and $r \in \{m+1, \ldots, n-1\}$. In particular, these vertices are given by $\tilde{q}^{sr}$ defined in equation (17).

Hence, all the vertices of the polytopes are defined by $\tilde{q}^{sr}$, $s \in \{1, \ldots, m\}$, $r \in \{m+1, \ldots, n\}$. Therefore, the set of vertices of the polygon generated by the points in the set Q is $\tilde{Q}$. ∎

Before we state the solution to the optimization problem (15), we define the following:

$$(s^*, r^*) \in \operatorname*{argmin}_{\substack{r \in \{m+1, \ldots, n\} \\ s \in \{1, \ldots, m\}}} \frac{(I_r^0 - I_r^1)T_s - (I_s^0 - I_s^1)T_r}{I_s^1 I_r^0 - I_s^0 I_r^1}, \quad \text{and}$$

$$g_{\text{two-sensors}}(s^*, r^*) = \frac{(I_{r^*}^0 - I_{r^*}^1)T_{s^*} - (I_{s^*}^0 - I_{s^*}^1)T_{r^*}}{I_{s^*}^1 I_{r^*}^0 - I_{s^*}^0 I_{r^*}^1}.$$

We also define

$$w^* = \operatorname*{argmin}_{w \in \{1, \ldots, n\}} \max \left\{ \frac{T_w}{I_w^0}, \frac{T_w}{I_w^1} \right\}, \quad \text{and}$$

$$g_{\text{one-sensor}}(w^*) = \max \left\{ \frac{T_{w^*}}{I_{w^*}^0}, \frac{T_{w^*}}{I_{w^*}^1} \right\}.$$

Theorem 2 (Worst case optimization): For the optimization problem (15), an optimal probability vector is given by:

$$q^* = \begin{cases} e_{w^*}, & \text{if } g_{\text{one-sensor}}(w^*) \leq g_{\text{two-sensors}}(s^*, r^*), \\ \tilde{q}^{s^* r^*}, & \text{if } g_{\text{one-sensor}}(w^*) > g_{\text{two-sensors}}(s^*, r^*), \end{cases}$$

and the minimum value of the function is given by:

$$\min \left\{ g_{\text{one-sensor}}(w^*), g_{\text{two-sensors}}(s^*, r^*) \right\}.$$

Proof: We invoke Lemma 6, and note that a minimum should lie at some vertex of the simplex Δ_{n-1}, or at some point in the set Q. Note that $g^0 = g^1$ on the set Q, hence the problem of minimizing $\max\{g^0, g^1\}$ reduces to minimizing g^0 on the set Q. From Theorem 1, we know that g^0 achieves the minima at some extreme point of the feasible region. From Lemma 7, we know that the vertices of the polytope generated by points in set Q are given by set $\tilde{Q}$. We further note that $g_{\text{two-sensors}}(s, r)$ and $g_{\text{one-sensor}}(w)$ are the value of objective function at the points in the set $\tilde{Q}$ and the vertices of the probability simplex Δ_{n-1} respectively, which completes the proof. ∎

C. Scenario III (Optimization of the average decision time):

For the multi-objective optimization problem of minimizing all the decision times simultaneously on the simplex, we formulate the single aggregate objective function as the average of these decision times. The resulting optimization problem, for $M \geq 2$, is posed in the following way:

$$\begin{aligned} \text{minimize} \quad & \frac{1}{M}(g^0(q) + \ldots + g^{M-1}(q)), \\ \text{subject to} \quad & q \in \Delta_{n-1}. \end{aligned} \tag{21}$$

In the following discussion we assume $n > M$, unless otherwise stated. We analyze the optimization problem in equation (21) as follows:

Lemma 8 (Non-vanishing Jacobian): The objective function in optimization problem in equation (21) has no critical point on Δ_{n-1} if the vectors $T, I^0, \ldots, I^{M-1} \in \mathbb{R}_{>0}^n$ are linearly independent.

Proof: The Jacobian of the objective function in the optimization problem in equation (21) is

$$\frac{1}{M} \frac{\partial}{\partial q} \sum_{k=0}^{M-1} g^k = \Gamma \psi(q),$$

where $\Gamma = \frac{1}{M} \begin{bmatrix} T & -I^0 & \cdots & -I^{M-1} \end{bmatrix} \in \mathbb{R}^{n \times (M+1)}$, and $\psi : \Delta_{n-1} \to \mathbb{R}^{M+1}$ is defined by

$$\psi(q) = \left[\sum_{k=0}^{M-1} \frac{1}{q \cdot I^k} \quad \frac{q \cdot T}{(q \cdot I^0)^2} \quad \cdots \quad \frac{q \cdot T}{(q \cdot I^{M-1})^2} \right]^{\mathsf{T}}.$$

For $n > M$, if the vectors $T, I^0, \ldots, I^{M-1}$ are linearly independent, then Γ is full rank. Further, the entries of ψ are non-zero on the probability simplex Δ_{n-1}. Hence, the Jacobian does not vanish anywhere on the probability simplex Δ_{n-1}. ∎

Lemma 9 (Case of Independent Information): For $M = 2$, if I^0 and I^1 are linearly independent, and $T = \alpha_0 I^0 + \alpha_1 I^1$, for some $\alpha_0, \alpha_1 \in \mathbb{R}$, then the following statements hold:

i) if α_0 and α_1 have opposite signs, then $g^0 + g^1$ has no critical point on the simplex Δ_{n-1}, and

ii) for $\alpha_0, \alpha_1 > 0$, $g^0 + g^1$ has a critical point on the simplex Δ_{n-1} if and only if there exists $v \in \Delta_{n-1}$ perpendicular to the vector $\sqrt{\alpha_0} I^0 - \sqrt{\alpha_1} I^1$.

Proof: We notice that the Jacobian of $g^0 + g^1$ satisfies

$$\begin{aligned} (q \cdot I^0)^2 (q \cdot I^1)^2 & \frac{\partial}{\partial q}(g^0 + g^1) \\ &= T \left((q \cdot I^0)(q \cdot I^1)^2 + (q \cdot I^1)(q \cdot I^0)^2 \right) \\ &\quad - I^0(q \cdot T)(q \cdot I^1)^2 - I^1(q \cdot T)(q \cdot I^0)^2. \end{aligned} \tag{22}$$

Substituting $T = \alpha_0 I^0 + \alpha_1 I^1$, equation (22) becomes

$$\begin{aligned} (q \cdot I^0)^2 & (q \cdot I^1)^2 \frac{\partial}{\partial q}(g^0 + g^1) \\ &= \left(\alpha_0 (q \cdot I^0)^2 - \alpha_1 (q \cdot I^1)^2 \right) \left((q \cdot I^1)I^0 - (q \cdot I^0)I^1 \right). \end{aligned}$$

Since I^0, and I^1 are linearly independent, we have

$$\frac{\partial}{\partial q}(g^0 + g^1) = 0 \iff \alpha_0 (q \cdot I^0)^2 - \alpha_1 (q \cdot I^1)^2 = 0.$$

Hence, $g^0 + g^1$ has a critical point on the simplex Δ_{n-1} if and only if

$$\alpha_0 (q \cdot I^0)^2 = \alpha_1 (q \cdot I^1)^2. \tag{23}$$

Notice that, if α_0, and α_1 have opposite signs, then equation (23) can not be satisfied for any $q \in \Delta_{n-1}$, and hence, $g^0 + g^1$ has no critical point on the simplex Δ_{n-1}.

If $\alpha_0, \alpha_1 > 0$, then equation (23) leads to

$$q \cdot (\sqrt{\alpha_0} I^0 - \sqrt{\alpha_1} I^1) = 0.$$

Therefore, $g^0 + g^1$ has a critical point on the simplex Δ_{n-1} if and only if there exists $v \in \Delta_{n-1}$ perpendicular to the vector $\sqrt{\alpha_0} I^0 - \sqrt{\alpha_1} I^1$. ∎

Lemma 10 (Optimal number of sensors): For $n > M$, if each $(M+1) \times (M+1)$ submatrix of the matrix

$$\Gamma = \begin{bmatrix} T & -I^0 & \cdots & -I^{M-1} \end{bmatrix} \in \mathbb{R}^{n \times (M+1)}$$

is full rank, then the following statements hold:

i) every solution of the optimization problem (21) lies on the probability simplex $\Delta_{M-1} \subset \Delta_{n-1}$; and

ii) every time-optimal policy requires at most M sensors to be observed.

Proof: From Lemma 8, we know that if $T, I^0, \ldots, I^{M-1}$ are linearly independent, then the Jacobian of the objective function in equation (21) does not vanish anywhere on the simplex Δ_{n-1}. Hence, a minimum lies at some simplex Δ_{n-2}, which is the boundary of the simplex Δ_{n-1}. Notice that, if $n > M$ and the condition in the lemma holds, then the projections of $T, I^0, \ldots, I^{M-1}$ on the simplex Δ_{n-2} are also linearly independent, and the argument repeats. Hence, a minimum lies at some simplex Δ_{M-1}, which implies that optimal policy requires at most M sensors to be observed. ∎

Lemma 11 (Optimization on an edge): Given two vertices e_s and e_r, $s \neq r$, of the probability simplex Δ_{n-1}, then for the objective function in the problem (21) with $M = 2$, the following statements hold:

i) if $g^0(e_s) < g^0(e_r)$, and $g^1(e_s) < g^1(e_r)$, then the minima, along the edge joining e_s and e_r, lies at e_s, and optimal value is given by $\frac{1}{2}(g^0(e_s) + g^1(e_s))$; and

ii) if $g^0(e_s) > g^0(e_r)$, and $g^1(e_s) < g^1(e_r)$, or vice versa, then the minima, along the edge joining e_s and e_r, lies at the point $q^* = (1 - t^*)e_s + t^* e_r$, where

$$t^* = \frac{1}{1+\mu} \in \,]0, 1[,$$

$$\mu = \frac{I_r^0 \sqrt{T_s I_r^1 - T_r I_s^1} - I_r^1 \sqrt{T_r I_s^0 - T_s I_r^0}}{I_s^1 \sqrt{T_r I_s^0 - T_s I_r^0} - I_s^0 \sqrt{T_s I_r^1 - T_r I_s^1}} > 0,$$

and the optimal value is given by

$$\frac{1}{2}(g^0(q^*) + g^1(q^*))$$

$$= \frac{1}{2} \left(\sqrt{\frac{T_s I_r^1 - T_r I_s^1}{I_s^0 I_r^1 - I_r^0 I_s^1}} + \sqrt{\frac{T_r I_s^0 - T_s I_r^0}{I_s^0 I_r^1 - I_r^0 I_s^1}} \right)^2.$$

Proof: We observe from Lemma 5 that both g^0, and g^1 are monotonically non-increasing or non-decreasing along any line. Hence, if $g^0(e_s) < g^0(e_r)$, and $g^1(e_s) < g^1(e_r)$, then the minima should lie at e_s. This concludes the proof of the first statement. We now establish the second statement. We note that any point on the line segment connecting e_s and e_r can be written as $q(t) = (1-t)e_s + t e_r$. The value of $g^0 + g^1$ at q is

$$g^0(q(t)) + g^1(q(t)) = \frac{(1-t)T_s + tT_r}{(1-t)I_s^0 + tI_r^0} + \frac{(1-t)T_s + tT_r}{(1-t)I_s^1 + tI_r^1}.$$

Differentiating with respect to t, we get

$$g^{0'}(q(t)) + g^{1'}(q(t))$$
$$= \frac{I_s^0 T_r - T_s I_r^0}{(I_s^0 + t(I_r^0 - I_s^0))^2} + \frac{I_s^1 T_r - T_s I_r^1}{(I_s^1 + t(I_r^1 - I_s^1))^2}. \quad (24)$$

Notice that the two terms in equation (24) have opposite sign. Setting the derivative to zero, and choosing the value of t in $[0, 1]$, we get $t^* = \frac{1}{1+\mu}$, where μ is as defined in the statement of the theorem. The optimal value of the function can be obtained, by substituting $t = t^*$ in the expression for $\frac{1}{2}(g^0(q(t)) + g^1(q(t)))$. ∎

Theorem 3 (Optimization of average decision time): For the optimization problem in equation (21) with $M = 2$, the following statements hold:

i) If I^0, I^1 are linearly dependent, then the solution lies at some vertex of the simplex Δ_{n-1}.

ii) If I^0 and I^1 are linearly independent, and $T = \alpha_0 I^0 + \alpha_1 I^1$, $\alpha_0, \alpha_1 \in \mathbb{R}$, then the following statements hold:

   a) If α_0 and α_1 have opposite signs, then the optimal solution lies at some edge of the simplex Δ_{n-1}.

   b) If $\alpha_0, \alpha_1 > 0$, then the optimal solution may lie in the interior of the simplex Δ_{n-1}.

iii) If every 3×3 sub-matrix of the matrix $\begin{bmatrix} T & I^0 & I^1 \end{bmatrix} \in \mathbb{R}^{n \times 3}$ is full rank, then a minimum lies at an edge of the simplex Δ_{n-1}.

Proof: We start by establishing the first statement. Since, I^0 and I^1 are linearly dependent, there exists a $\gamma > 0$ such that $I^0 = \gamma I^1$. For $I^0 = \gamma I^1$, we have $g^0 + g^1 = (1 + \gamma)g^0$. Hence, the minima of $g^0 + g^1$ lies at the same point where g^0 achieves the minima. From Theorem 1, it follows that g^0 achieves the minima at some vertex of the simplex Δ_{n-1}.

To prove the second statement, we note that from Lemma 9, it follows that if α_0, and α_1 have opposite signs, then the Jacobian of $g^0 + g^1$ does not vanish anywhere on the simplex Δ_{n-1}. Hence, the minima lies at the boundary of the simplex. Notice that the boundary, of the simplex Δ_{n-1}, are n simplices Δ_{n-2}. Notice that the argument repeats till $n > 2$. Hence, the optima lie on one of the $\binom{n}{2}$ simplices Δ_1, which are the edges of the original simplex. Moreover, we note that from Lemma 9, it follows that if $\alpha_0, \alpha_1 > 0$, then we can not guarantee the number of optimal set of sensors. This concludes the proof of the second statement.

To prove the last statement, we note that it follows immediately from Lemma 10 that a solution of the optimization problem in equation (21) would lie at some simplex Δ_1, which is an edge of the original simplex. ∎

Note that, we have shown that, for $M = 2$ and a generic set of sensors, the solution of the optimization problem in equation (21) lies at an edge of the simplex Δ_{n-1}. The optimal value of the objective function on a given edge was determined in Lemma 11. Hence, an optimal solution of this problem can be determined by a comparison of the optimal values at each edge.

For the multiple hypothesis case, we have determined the time-optimal number of the sensors to be observed in Lemma 10. In order to identify these sensors, one needs to solve the optimization problem in equation (21). We notice that the objective function in this optimization problem is non-convex, and is hard to tackle analytically for $M > 2$. Interested reader may refer to some efficient iterative algorithms in linear-fractional programming literature (e.g., [6]) to solve these problems.

VI. NUMERICAL EXAMPLES

We consider four sensors connected to a fusion center. We assume that the sensors take binary measurements. The probabilities of their measurement being zero, under two hypotheses, and their processing times are given in the Table I.

TABLE I
CONDITIONAL PROBABILITIES OF MEASUREMENT BEING ZERO

Sensor	Probability(0)		Processing Time
	Hypothesis 0	Hypothesis 1	
1	0.4076	0.5313	0.6881
2	0.8200	0.3251	3.1960
3	0.7184	0.1056	5.3086
4	0.9686	0.6110	6.5445

We performed Monte-Carlo simulations with the mis-detection and false-alarm probabilities fixed at 10^{-3}, and computed expected decision times. In Table II, the numerical expected decision times are compared with the decision times obtained analytically in equation (7). The difference in the numerical and the analytical decision times is explained by the Wald's asymptotic approximations.

TABLE II
DECISION TIMES FOR VARIOUS SENSOR SELECTION PROBABILITIES

Sensor selection probability	Expected Decision Time			
	Hypothesis 0		Hypothesis 1	
	Analytical	Numerical	Analytical	Numerical
[1,0,0,0]	154.39	156.96	152.82	157.26
[0.3768,0,0,0.6232]	124.35	129.36	66.99	76.24
[0.25,0.25,0.25,0.25]	55.04	59.62	50.43	55.73

In the Table III, we compare optimal policies in each Scenario I, II, and III with the policy when each sensor is chosen uniformly. It is observed that the optimal policy improves the expected decision time significantly over the uniform policy.

TABLE III
COMPARISON OF THE UNIFORM AND OPTIMAL POLICIES

Scenario	Uniform policy	Optimal policy	
	Objective function	Optimal probability vector	Optimal objective function
I	59.62	[0,0,1,0]	41.48
II	55.73	[0,0,1,0]	45.24
III	57.68	[0,0.4788,0.5212,0]	43.22

We performed another set of simulations for the multi-hypothesis case. We considered a ternary detection problem, where the underlying signal $x = 0, 1, 2$ needs to be detected from the available noisy data. We considered a set of four sensors and their conditional probability distribution is given in Tables IV and V. The processing time of the sensors were chosen to be the same as in Table I.

The set of optimal sensors were determined for this set of data. Monte-Carlo simulations were performed with the thresholds $\eta_k, k \in \{0, \ldots, M-1\}$ set at 10^{-6}. A comparison of the uniform sensor selection policy and an optimal sensor

TABLE IV
CONDITIONAL PROBABILITIES OF MEASUREMENT BEING ZERO

Sensor	Probability(0)		
	Hypothesis 0	Hypothesis 1	Hypothesis 2
1	0.4218	0.2106	0.2769
2	0.9157	0.0415	0.3025
3	0.7922	0.1814	0.0971
4	0.9595	0.0193	0.0061

TABLE V
CONDITIONAL PROBABILITIES OF MEASUREMENT BEING ONE

Sensor	Probability(1)		
	Hypothesis 0	Hypothesis 1	Hypothesis 2
1	0.1991	0.6787	0.2207
2	0.0813	0.7577	0.0462
3	0.0313	0.7431	0.0449
4	0.0027	0.5884	0.1705

selection policy is presented in Table VI. Again, the significant difference between the average decision time in the uniform and the optimal policy is evident.

TABLE VI
COMPARISONS OF THE UNIFORM AND AN OPTIMAL POLICY

Policy	Selection Probability	Average Decision Time	
	q^*	Analytical	Numerical
Optimal	[0, 0.9876, 0, 0.0124]	38.57	42.76
Uniform	[0.25, 0.25, 0.25, 0.25]	54.72	54.14

We note that the optimal results, we obtained, may only be sub-optimal because of the asymptotic approximations in equations (3) and (5). We further note that, for small error probabilities and large sample sizes, these asymptotic approximations yield fairly accurate results [5], and in fact, this is the regime in which it is of interest to minimize the expected decision time. Therefore, for all practical purposes the obtained optimal scheme is very close to the actual optimal scheme.

VII. CONCLUSIONS

In this paper, we considered the problem of sequential decision making. We developed versions SPRT and MSPRT where the sensor switches at each observation. We used these sequential procedures to decide reliably. We found out the set of optimal sensors to decide in minimum time. For the binary hypothesis case, three performance metrics were considered and it was found that for a generic set of sensors at most two sensors are optimal. Further, it was shown that for M underlying hypotheses, and a generic set of sensors, an optimal policy requires at most M sensors to be observed. A procedure for identification of the optimal sensor was developed. In the binary hypothesis case, the computational complexity of the procedure for the three scenarios, namely, the conditioned decision time, the worst case decision time, and the average decision time, was $\mathcal{O}(n)$, $\mathcal{O}(n^2)$, and $\mathcal{O}(n^2)$, respectively.

Many further extensions to the results presented here are possible. First, the time-optimal scheme may not be energy optimal. In particular, the time optimal set of sensors may be

the most distant sensors from the fusion center. Given that the power to transmit the signal to the fusion center is proportional to the distance from the fusion center, the time-optimal scheme is no where close to the energy optimal scheme. This trade off can be taken care of by adding a term proportional to distance in the objective function.

When we choose only one or two sensors every time, issues of robustness do arise. In case of sensor failure, we need to determine the next best sensor to switch. A list of sensors, with increasing decision time, could be prepared beforehand and in case of sensor failure, the fusion center should switch to the next best set of sensors.

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